Launch Dynamics and Gravity Turn Maneuvers

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This brief set of notes introduces the dynamics of launch systems and illustrates how these dynamics can be simulated to predict a rocket trajectory. We will explicitly treat the simplified problem where the Earth is assumed to be spherical and non-rotating. In this case, the simplest maneuver needed to achieve orbit is a single pitch-over maneuver, where the launch vehicle is given some component of velocity parallel to the surface of the Earth. In this case the trajectory is planar, and the dynamics can be captured in two dimensions. Once this is outlined, I will then briefly discuss launches in three dimensions and outline how the equations of motion need to be modified in order to include the Earths rotation.

I. LAUNCH – TWO DIMENSIONAL SIMPLIFIED MODEL

In the approximation that the Earth is non-rotating, a launch to orbit can be effectively described in two dimensions (*i.e.*, vertical direction, and "down-range" direction). Let us choose a representation for the full *state* of the the launch vehicle. In two-dimensions, this must be parameterized by four quantities; in many cases it is common at this point to choose our representation as the cartesian components of position and velocity, $\{x(t), y(t), v_x(t), v_y(t)\}$. For an orbital launch vehicle, it is in fact more convenient to choose the following representation:

$$\begin{aligned} r(t) &\equiv \text{Radial position of vehicle} \\ \theta(t) &\equiv \text{Declination angle of the vehicle from the origin} \\ v(t) &\equiv \text{Magnitude of the vehicle's velocity} \\ \gamma(t) &\equiv \text{Flight path angle of the vehicle,} \end{aligned}$$

where $r = |\boldsymbol{r}|$ and $v = |\boldsymbol{v}|$. These state components are illustrated in Figure 1; the rocket-fixed (non-inertial) coordinate system is defined by the orthonormal vectors $\hat{\boldsymbol{u}}_t$ (tangent to velocity) and $\hat{\boldsymbol{u}}_n$ (normal to velocity). We refer to this rocket-fixed reference frame as $\mathscr{L}$ and the Earth-fixed (pseudo-inertial) reference frame as

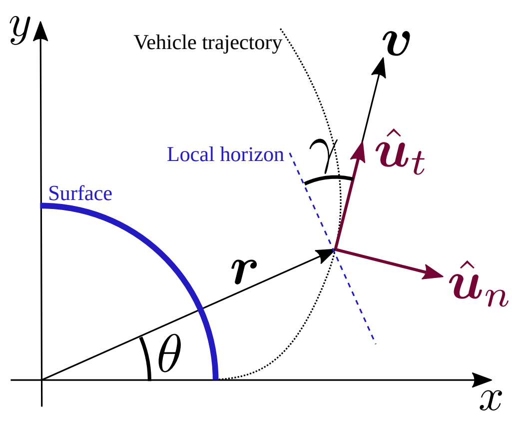

FIG. 1: Earth-fixed and rocket-fixed coordinate frames in 2D, with rocket coordinates either given by $\{\boldsymbol{r}, \boldsymbol{v}\}$, or by $\{r, \theta, v, \gamma\}$.

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$\mathscr{N}$. In terms of the rotating reference frame, we can express the velocity and acceleration of the vehicle as

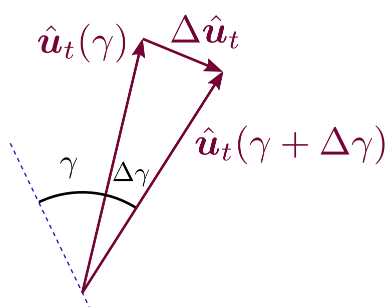
$$\boldsymbol{v} = v\hat{\boldsymbol{u}}_t \tag{1.1}$$

$$\boldsymbol{a} = \left(\frac{d\boldsymbol{v}}{dt}\right)_{\mathscr{N}} = \left(\frac{d\boldsymbol{v}}{dt}\right)_{\mathscr{L}} + \boldsymbol{\omega} \times \boldsymbol{v} \tag{1.2}$$

The first term in the acceleration is

$$\left(\frac{d\boldsymbol{v}}{dt}\right)_{\mathscr{L}} = \frac{d}{dt}(v\hat{\boldsymbol{u}}_t) = \dot{v}\hat{\boldsymbol{u}}_t + v\dot{\hat{\boldsymbol{u}}}_t = \dot{v}\hat{\boldsymbol{u}}_t + v\frac{d\gamma}{dt}\frac{d\hat{\boldsymbol{u}}_t}{d\gamma},$$

where in the last equality we have utilized the chain rule. The derivative $d\hat{\boldsymbol{u}}_t/d\gamma$ is found by considering the following figure (*c.f.*, Figure 1):



By taking the limit $\Delta\gamma \to 0$ in this figure, it is clear that $\Delta\hat{\boldsymbol{u}}_t$ is orthogonal to $\hat{\boldsymbol{u}}_t$, so that $d\hat{\boldsymbol{u}}_t/d\gamma = \hat{\boldsymbol{u}}_n$, and

$$\left(\frac{d\boldsymbol{v}}{dt}\right)_{\mathscr{L}} = \dot{v}\hat{\boldsymbol{u}}_t + v\dot{\gamma}\hat{\boldsymbol{u}}_n, \tag{1.3}$$

Moreover, since the angular velocity of the $\mathscr{L}$ with respect to $\mathscr{N}$ is $\boldsymbol{\omega} = \dot{\theta}\hat{\boldsymbol{u}}_z$, with $\hat{\boldsymbol{u}}_z$ pointing out of the page, we have

$$\boldsymbol{\omega} \times \boldsymbol{v} = \dot{\theta}v(\hat{\boldsymbol{u}}_z \times \hat{\boldsymbol{u}}_t) = -\dot{\theta}v\hat{\boldsymbol{u}}_n = -\frac{v^2\cos\gamma}{r}\hat{\boldsymbol{u}}_n \tag{1.4}$$

Utilizing Eqs. (1.3) and (1.4) in Eq. (1.2), the acceleration of the rocket can be written as

$$\boldsymbol{a} = \dot{v}\hat{\boldsymbol{u}}_t + \left(v\dot{\gamma} - \frac{v^2\cos\gamma}{r}\right)\hat{\boldsymbol{u}}_n \tag{1.5}$$

Equations (1.1) and (1.5) describe the *kinematics* of the rocket in the $\mathscr{L}$ reference frame. The *dynamics* are given by Newton's second law, which can be decomposed into $\mathscr{L}$-frame components as

$$F_t = ma_t = m\dot{v}$$
$$F_n = ma_n = m\left(v\dot{\gamma} - \frac{v^2\cos\gamma}{r}\right).$$

The primary forces acting on the rocket are gravity, atmospheric drag, thrust, and lift. We can assume the force of lift is in the purely normal direction, and that the thrust can be gimbaled about some angle α about the tangential axis. In practice, the thrust and lift will be applied at locations on the vehicle which differ from the vehicle's center of mass. In these cases, the parallel axis theorem can be applied to determine the action of the thrust and lift on the center of mass. For simplicity, however, we will for now treat our vehicle as a point mass. The forces then acting on the rocket can then be written explicitly as

$$\begin{aligned}
\boldsymbol{F}_{\text{grav}} &= -m\frac{\mu}{r^2}\hat{\boldsymbol{r}} = -m\frac{\mu}{r^2}(\sin\gamma\hat{\boldsymbol{u}}_t + \cos\gamma\hat{\boldsymbol{u}}_n) \\
\boldsymbol{F}_{\text{drag}} &= -\frac{1}{2}AC_D\rho v^2\hat{\boldsymbol{u}}_t \\
\boldsymbol{F}_{\text{lift}} &= \frac{1}{2}AC_L\rho v^2\hat{\boldsymbol{u}}_n \\
\boldsymbol{F}_{\text{thrust}} &= T(\cos\alpha\hat{\boldsymbol{u}}_t + \sin\alpha\hat{\boldsymbol{u}}_n)
\end{aligned}$$

The dynamical equations now take the form

$$F_t = m\dot{v} = T\cos\alpha - \frac{1}{2}AC_D\rho v^2 - m\frac{\mu}{r^2}\sin\gamma \tag{1.6a}$$

$$F_n = m\left(v\dot{\gamma} - \frac{v^2\cos\gamma}{r}\right) = T\sin\alpha + \frac{1}{2}AC_L\rho v^2 - m\frac{\mu}{r^2}\cos\gamma \tag{1.6b}$$

The first-order dynamical equations for our state parameters $\{r, \theta, v, \gamma\}$ are now given by

$$\dot{r} = v\sin\gamma \tag{1.7}$$

$$\dot{\theta} = \frac{v\cos\gamma}{r} \tag{1.8}$$

$$\dot{v} = \frac{T}{m}\cos\alpha - \frac{1}{2m}AC_D\rho v^2 - \frac{\mu}{r^2}\sin\gamma \tag{1.9}$$

$$\dot{\gamma} = \frac{T}{vm}\sin\alpha + \frac{1}{2m}AC_L\rho v - \frac{1}{v}\frac{\mu}{r^2}\cos\gamma + \frac{v\cos\gamma}{r}, \tag{1.10}$$

where the first two equations are determined purely from the geometry, and the latter two equations are determined from Eqs. (1.6). In addition to these equations, the mass of the vehicle evolves in time as propellant is expended from the vehicle. This can be expressed in terms of the rocket engine's specific impulse,

$$\dot{m} = -\frac{T}{g_0 I_{\text{sp}}} \tag{1.11}$$

If a constant thrust is assumed, this equation can be directly integrated to determine the burn time for the engines. During the burn, the above equation integrates to

$$m(t) = m_0 - \frac{T}{g_0 I_{\text{sp}}}t. \tag{1.12}$$

Taking $m(t_f) = m_0 - m_p$, where m_0 is the initial mass of the vehicle (vehicle *plus* propellant) and m_p is the mass of the propellant, then

$$m_p = \frac{T}{g_0 I_{\text{sp}}}t_f \quad \Rightarrow \quad t_f = \frac{m_p g_0 I_{\text{sp}}}{T} \tag{1.13}$$

determines the burn time t_f for the engines.

The above system of equations can now be numerically integrated to provide the trajectory. The gimbal angle $\alpha = \alpha(t)$ is input from the guidance system, and is typically chosen to optimize the trajectory in terms of aerodynamic stresses, fuel cost, and trajectory requirements. The atmospheric density can be approximated as $\rho(r) = \rho_0 \exp(-(r - R_E)/h_0)$, with $\rho_0 = 1.225$ kg/m^3 the atmospheric density at sea level, $h_0 = 7.5$ km a scale altitude, and $(r - R_E)$ the altitude of the vehicle above sea level. The coefficients C_D and C_L are dependent on the geometry of the vehicle. Lastly, if constant thrust is assumed,

$$T(t) = \begin{cases} T & 0 \le t \le t_f, \\ 0 & t > t_f. \end{cases} \tag{1.14}$$

This latter assumption of constant thrust can easily be relaxed, but in these cases a more accurate relationship between T and $\dot{m}$ is needed.

Gravity Turn Maneuvers

Let us consider a simple example in which lift forces are nulled or negligible, and where the gimbal angle α is constant and set to zero. In this case the equations of motion reduce to

$$\dot{r} = v \sin\gamma \tag{1.15a}$$

$$\dot{\theta} = \frac{v\cos\gamma}{r} \tag{1.15b}$$

$$\dot{v} = \frac{T}{m} - \frac{1}{2m} A C_D \rho v^2 - \frac{\mu}{r^2}\sin\gamma \tag{1.15c}$$

$$\dot{\gamma} = -\frac{1}{v}\frac{\mu}{r^2}\cos\gamma + \frac{v\cos\gamma}{r} \tag{1.15d}$$

where m is given by

$$m(t) = \begin{cases} m_0 - \frac{T}{g_0 I_{\text{sp}}} t & \text{for } 0 < t < t_f, \\ m_0 - m_p & \text{for } t > t_f, \end{cases} \tag{1.16}$$

and where T is given by Eq. (1.14). Note that in order to integrate Eqs. (1.15). we must assume some small non-zero velocity in order to prevent the equations of motion from becoming singular; this is simply an artifact of our chosen coordinate system, as $\hat{\boldsymbol{u}}_t$ and $\hat{\boldsymbol{u}}_n$ are undefined for a stationary vehicle.

By inspection of the equation of motion for flight-path angle (Eq. (1.15d)) it can be seen that, given some initial *non-vertical* flight-path angle $\gamma < 90°$, the flight-path angle will decrease with time. Since for an orbital trajectory we eventually require the velocity to be mainly horizontal with respect to the Earth's surface ($\gamma \approx 0$), we can use the natural dynamics in Eq. (1.15d) to "pull" the trajectory towards a horizontal path. This is known as a *gravity turn* maneuver. Note that the dynamics of $\gamma(t)$ giving rise to the gravity turn maneuver are completely intuitive: if one simply throws a ball at some non-vertical angle, the curved ballistic trajectory will also exhibit a flight-path angle which reduces with time. At the top of this balistic trajectory, the velocity is purely horizontal ($\gamma = 0$), and at any later time the velocity is horizontal and downward (corresponding to $\gamma < 0$). If, on the other hand, the ball is thrown exactly vertically, the flight path angle remains at $\gamma = 90°$ until the vehicle slows, stops, and begins to fall (at which point the flight-path angle instantaneously and discontinuously becomes $\gamma = -90°$). The only difference between a typical ballistic trajectory and a gravity turn maneuver is that, in the latter, the vehicle is being accelerated by the rocket's thrust, so that at some point the horizontal velocity can become large enough to achieve orbit.

In some sense, the gravity turn is an optimal way of achieving orbit since, throughout the trajectory, the thrust is colinear with the velocity and therefore is most efficiently used to achieve orbital velocity. If a nonzero gimbal angle is used to reduce γ for horizontal flight, then $\dot{v}$ will be reduced according to Eq. (1.15c). In reality, some small amount of gimbaling (or the equivalent of gimbaling achieved by control thrusters) will typically be used by the guidance system to more optimally reduce aerodynamic stresses and place the vehicle in the desired orbit.

The gravity turn maneuver outlined above assumes some initial non-vertical flight-path angle. As a slightly more realistic scenario, we can instead assume that the launch vehicle takes off vertically, and at some point early on in its trajectory a small nonzero flight-path angle is established (which can be achieved either through small control thrusters or some brief gimbaling of the rocket engines). In addition to being a more realistic simulation, it avoids the initial singularity at $v = 0$ as the initial equations of motion are given by

$$\dot{r} = v \tag{1.17a}$$

$$\dot{\theta} = 0 \tag{1.17b}$$

$$\dot{v} = \frac{T}{m} - \frac{1}{2m} A C_D \rho v^2 - \frac{\mu}{r^2} \tag{1.17c}$$

$$\dot{\gamma} = 0 \tag{1.17d}$$

$$\dot{m} = -\frac{T}{g_0 I_{\text{sp}}} \tag{1.17e}$$

We can therefore simulate a launch by integrating Eqs. (1.17) up to the point of the pitch-over maneuver, then integrating Eqs. (1.15) for the remainder of the trajectory. Equivalently, we can simply integrate Eq. (1.15) with initial conditions

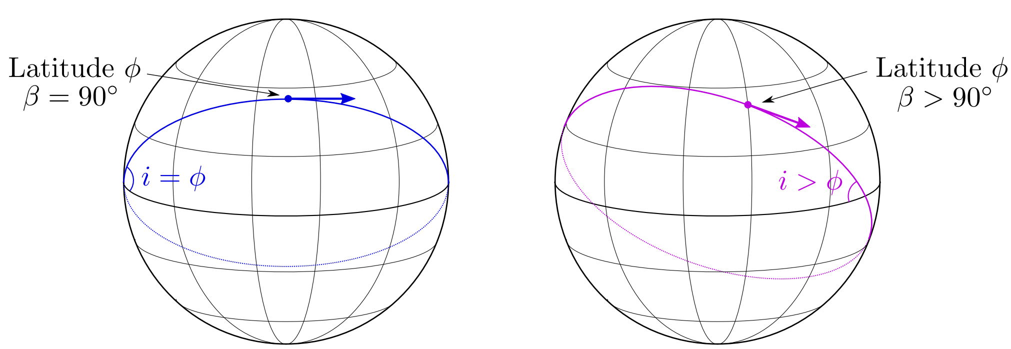
$$r(0) = R_E, \quad \theta(0) = 0^\circ, \quad v(0) = 0 \text{ m/s}, \quad \gamma(0) = 90^\circ, \tag{1.18}$$

then at the time of pitch-over, set $\gamma(t_{\text{po}}) = \gamma_{\text{po}}$. Since initially $\gamma(0) = 90^\circ$, it will remain at $\gamma(t) = 90^\circ$ until pitch-over (since in this case $\dot{\gamma} = 0$). Once pitch-over angle is set to γ_{po} at $t = t_{\text{po}}$ it will then evolve according to Eq. (1.15d).

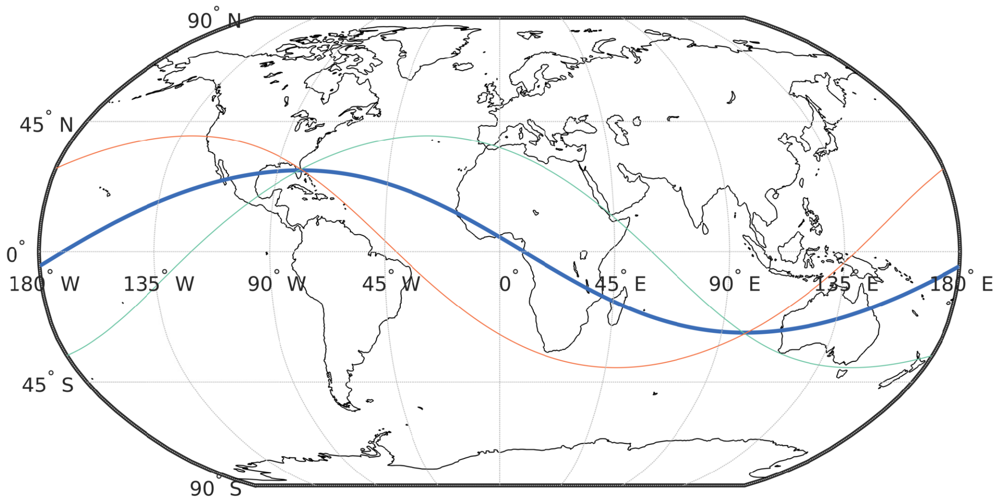
II. LAUNCHES IN THREE DIMENSIONS

When launching in three dimensions, one must take into account both the *launch azimuth* β (the cardinal direction of the pitch-over) and the launch site latitude ϕ. The relationship between latitude ϕ and launch azimuth β determines the inclination of the resulting orbit, assuming that a single pitch-over, direct-to-orbit launch is performed. For any launch from a geographic latitude ϕ, the achievable inclination of the resulting orbit is $i \geq \phi$.

This can be understood as follows: imagine that a spacecraft is in a circular orbit with instantaneous latitude ϕ, and has a velocity purely in the eastward direction. If we draw a great circle about the Earth intersecting this point, we realize that at this point the spacecraft has achieved its largest latitude, and it's latitude will begin to decrease. This implies that the inclination of the orbit is $i = \phi$, and also implies that the nodal crossing is -90° from this point. On the other hand, if the instantaneous velocity is not purely eastward, then the spacecraft is not at its maximum latitude along its great circle, which implies that the inclination of this orbit is $i > \phi$. These two cases are illustrated below:



In both cases, if we assume a non-rotating Earth, the instantaneous velocity direction at latitude ϕ corresponds to the launch azimuth direction β for a single pitch-over maneuver. In the case at left (purely eastward instantaneous velocity), we have $\beta = 90^\circ$, while in the case at right (southerly instantaneous velocity), we have $\beta > 90^\circ$. From these two cases, it becomes clear that for a direct launch-to-orbit from latitude ϕ, the *minimum* inclination one can achieve is $i = \phi$, and this minimum inclination is achieved by launching in a purely eastward direction. Another way of seeing this is by considering the ground tracks of orbits passing over the launch site. If we consider our launch site to be Cape Canveral ($\phi = 28^\circ$), we can plot the ground tracks for three different orbits with differing launch azimuths as follows:



We have assumed in this figure, for simplicity, purely circular orbits which have orbital period of 1 sidereal day (so that the ground track is a closed path). As can be seen from the figure, the orbit corresponding to a purely eastward launch velocity from Cape Canaveral (the thick, blue line) has the lowest orbital inclination of the three orbits, while the orbits corresponding to a north-east (green) and south-east (orange) launch azimuth have larger inclinations.

For a non-rotating and spherical Earth, it is straightforward to show from spherical trigonometry that

$$\cos i = \cos(\phi)\sin(\beta), \tag{2.1}$$

and moreover,

$$\sin\phi = \sin(i)\sin(\omega + \nu).$$

Note that Eq. (2.1) implies that $i \geq \phi$, agreeing with our previous geometric reasoning. Including the oblateness of the Earth alters Eq. (2.1), but does not change the fact that inclination is minimized when $\beta = 90°$.

Including the rotation of the Earth

The above argument establishing that $i \geq \phi$ does not rely on the assumption of a non-rotating Earth, as this rotation adds a purely eastward contribution to the velocity. However, this rotation will slightly alter Eq. (2.1), since for $\beta \neq 90°$ the inertial frame launch direction is not exactly equal to the Earth-fixed launch direction (the direction of the pitch-over as measured on the surface of the Earth). Specifically, its initial inertial velocity has an eastward component given by

$$\boldsymbol{v}_0 = \boldsymbol{\omega} \times \boldsymbol{r}_{\text{site}}$$

where $\boldsymbol{\omega}$ is the angular velocity of the Earth and $\boldsymbol{r}_{\text{site}}$ is the Earth-centered position vector of the launch site.

The problem becomes a bit more complicated, since the equations of motion now depend on both the inertial *and* the non-inertial (Earth-fixed) velocity. The three-dimensional drag and lift forces acting on the rocket depend on the relative velocity $\boldsymbol{v}_{\text{rel}}$ between the atmosphere and the rocket, which in first approximation (in absence of wind currents and assuming the atmosphere rotates with the Earth) is equal to

$$\boldsymbol{v}_{\text{rel}} = \boldsymbol{v} - \boldsymbol{\omega} \times \boldsymbol{r},$$

where $\boldsymbol{v}$ is the inertial velocity (integrated from $\boldsymbol{v}_0$ as given above) and $\boldsymbol{r}$ is the Earth-centered position of the rocket. The gravitational and thrust forces, however, must be computed as $\boldsymbol{F} = m\dot{\boldsymbol{v}}$. These additional details are straightforward to include in a numerical simulation, but require a full three-dimensional treatment.